ABSTRACT. This paper discusses dielectric properties of snow according to various dielectric models and compares them with experimental results. The complex permittivity of wet snow is assumed to consist of two parts, being the sum of the permittivity of dry snow (a mixture of ice and air) and the excess permittivity due to liquid water (resulting from the dielectric mixture of water and air). In particular the effect of liquid water is considered. Exponential models and structure-dependent models based on mixture theories by Taylor and Tinga and others are applied. It is shown that the assumption that water inclusions have the form of either randomly oriented discs or needles, or of spheres do, not get empirical confirmation but the inclusions are preferably prolate ellipsoids (ellipticity 0.16) or oblate ellipsoids (ellipticity 0.12), dry snow being a dielectric mixture of randomly oriented disc-shaped ice particles and air.

INTRODUCTION

The effect of liquid water upon the dielectric characteristics of snow has been a puzzle for glaciologists during the last few decades. Several papers dealing with the electrical properties of snow have been published. However, the results for wet snow are not always consistent with one another.

Snow can be treated as a three-component mixture consisting of air, ice, and water. A special case is dry snow which contains no liquid (free) water. The dielectric properties of these constituents are well known, including their frequency dependence (see, for example, Hallikainen, 1977). Experimental formulae for the complex permittivity of dry snow have also been presented (Nyfors, 1982; Tiuri and others, 1984). Numerous formulae explaining and predicting the dielectric characteristics of wet snow have been presented. These may be mixing formulae that have the permittivities of air, ice, and water as parameters, or they may even be linearized functions of density and wetness. More rigorous mixing theories take into account the microscopic structure of snow and the liquid water distribution. In this case, the resulting formulae usually contain additional parameters (for example, depolarization factors of the ice and water particles) that are determined by the type of snow. In this paper it is presumed that the effects of ice and water on the permittivity of wet snow can be superposed. In other words, the three-component mixing formula

$$\varepsilon_s = f(\varepsilon_i, \varepsilon_w)$$

is separable in the form

$$\varepsilon_s = f_1(\rho) + f_2(\rho, W)$$

where $\varepsilon_s$ is the complex permittivity of snow, $\rho$ is the dry density (see below), and $W$ is the wetness of snow. This means that wet snow is treated as a two-component mixture consisting of dry snow and liquid water. And dry snow is for its part a dielectric mixture of air and ice. There are papers employing this separation approach, for example Ambach and Denoth (1972), Denoth and Schinkelkopf (1978), Ambach and Denoth (1980), Matzler and others (1984), Tiuri and others (1984), see also Stiles and Ulaby (1981).

The first term in Equation (2) is the permittivity of dry snow, the density of which is equal to the "dry density" of the wet snow in question, i.e. the density which the snow sample would have if all the liquid water were replaced by air. Because the dielectric
properties of dry snow are fairly well known, the most interesting problem is the increase of the complex permittivity due to the free water, in other words the properties of dry snow are fairly well known, the most second term in Equation (2). The difference between the permittivities of wet snow (\(\varepsilon_w\)) and dry snow (\(\varepsilon_d\))

\[
\Delta \varepsilon = \varepsilon_w - \varepsilon_d = \Delta \varepsilon_w^* - j \Delta \varepsilon_w
\]

is therefore a result of the dielectric mixture of air and water. This mixture gives the increase of the real part of the permittivity of snow. And, because dry snow is practically lossless in comparison with water at microwave frequencies, the whole imaginary part of the dielectric constant of wet snow comes from this mixture.

DIELECTRIC PROPERTIES OF THE CONSTITUENTS

The relative permittivity of air is \(\varepsilon_{air} = 1 - j0\). At 1 GHz and 0°C the dielectric constant of ice is \(\varepsilon_{ice} = 3.15 - j0.003\). Air and ice are also quasi-dispersionless in the microwave range. Therefore the dielectric properties of dry snow obey the high-frequency approximation of the Debye equation, the relaxation frequency being less than 100 kHz. The real part of the permittivity is constant with frequency at microwave wavelengths, and, as has already been noted, the loss factor is negligible. Accordingly, the dispersion characteristics of wet snow originate from the frequency dependence of liquid water.

The complex permittivity of water \(\varepsilon_{w}\) follows fairly well the Debye equation

\[
\varepsilon_w = \varepsilon_{wm} + \frac{\varepsilon_{ws} - \varepsilon_{wm}}{1 + j \omega \tau_w} = \varepsilon_w^* - j \varepsilon_w
\]

where \(\varepsilon_{ws}\) is the static dielectric constant of water, \(\varepsilon_{wm}\) is the high-frequency dielectric constant of water, and \(\tau_w\) is the relaxation time of water. Experimental values for \(\varepsilon_{wm}\) range from 4.5 to 5.5. At 0°C the static permittivity is \(\varepsilon_{ws} = 88\) and the relaxation time \(\tau_w = 18\) ps.

Therefore the relaxation frequency is around 9 GHz under normal conditions. Hence most dielectric measurements of wet snow are carried out in the low-frequency range. Many snow sensors for determining the permittivity of snow operate near 1 GHz (Tiuri and others, 1982; Aebischer and Mütter, 1983; Denoth and others, 1984). In this paper the air-water mixture is treated wet snow as a three-component mixture in which the other component is air, \(\varepsilon_w\) is the complex permittivity of the inclusion material \(\varepsilon_i\) is the volume fraction of the inclusions.

\[
\Delta \varepsilon = \varepsilon_w^* - j \varepsilon_w
\]

Exponential models have been applied to dielectric mixtures by many authors:

\[
a = 1 \text{ Brown has presented a formula of this kind linearly dependent on the volume fraction (Wang and Schmugge, 1980). Fung (1982) has also applied it for determining the effective dielectric constant of a vegetated medium.}
\]

\[
a = 1/2 \text{ Birchak and others have derived this formula for the soil-water mixture (Birchak and others, 1974).}
\]

\[
a = 0.4 \text{ By treating wet snow as a three-component mixture, one can apply the exponential model to the classical results of Cumming (1952). The best fit is found by optimizing the parameter } a \text{, and the result is } a = 0.4.
\]

\[
a = 1/3 \text{ This value of the parameter } a \text{ will lead to a cubic function for the permittivity of two-phase mixtures.}
\]

The complex dielectric constant \(\varepsilon_r\) of a two-component mixture in which the other component is air, \(\varepsilon_{air}\), according to the exponential model, obeys the following formula (obtained directly from Equation (5)): 

\[
\varepsilon_r = (1 - \phi + \varepsilon_{air}^a) / a
\]

where \(\phi\) is the volume fraction of the inclusions. Therefore \(1 - \phi\) is the volume fraction occupied by air. \(\varepsilon_{air}\) is the complex permittivity of the inclusion material.

\[
\varepsilon_i = \varepsilon_i^* (1 - j \tan \delta_i) = \varepsilon_i^* (1 - j a \tan \delta_i)
\]

The following approximation, which assumes that the inclusion material has low losses, will be used:

\[
\varepsilon_i = \varepsilon_i^* (1 - j a \tan \delta_i) = \varepsilon_i^* (1 - j a \tan \delta_i)
\]

EXPERIMENTAL MODELS

Mixing formulae

The exponential models give the complex permittivity of the mixture \(\varepsilon_{mixx}\) as a function of the complex permittivities of its constituents \(\varepsilon_i\) through the equation

\[
\varepsilon_{mixx} = \sum f_i \varepsilon_i
\]

where \(f_i\) is the volume fraction of the \(i\)th constituent, and \(\Sigma f_i = 1\). The exponent \(a\) is the degree of the model, and \(0 < a < 1\).
The exponential mixing model can be applied to dry snow conceived as a dielectric mixture of air and ice (with the complex permittivity $3.15(1 - j0.001)$. When the parameter $a$ is $a = 1/2$, we have

$$\epsilon_d' = 1 + 1.69\rho_d' + 0.71\rho_d'^2$$  \hspace{1cm} (13)

$$\epsilon_d'' = (0.61\rho_d' + 0.52\rho_d'^2)\epsilon_{\text{ice}}$$  \hspace{1cm} (14)

which follows the experimental results fairly well.

The case $a = 1/3$ leads to the formula of Looyenga for dry snow (Stiles and Ulaby, 1981):

$$\epsilon_d' = (1 + 0.508\rho_d')^a,$$  \hspace{1cm} (15)

which explains the real part of the permittivity satisfactorily.

**Wet snow**

From Equations (9) and (10) the excess permittivity of wet snow due to the liquid water follows:

$$\Delta\epsilon_s' = \epsilon_{m}' - 1,$$  \hspace{1cm} (16)

$$\Delta\epsilon_s'' = \epsilon_{m}''.$$  \hspace{1cm} (17)

This is the difference between the permittivities of wet snow and the corresponding dry snow (that snow which is left when all liquid water is replaced by air).

Different values for the parameter $a$ give at 1 GHz and 0°C:

$$a = 1 \quad \Delta\epsilon_s' = 87.0W, \quad \epsilon_s'' = 0.79W,$$  \hspace{1cm} (18)

$$a = 1/2 \quad \Delta\epsilon_s' = 6.8W + 70.2W^2, \quad \epsilon_s'' = 1.04W + 8.72W^2,$$  \hspace{1cm} (19)

$$a = 0.4 \quad \Delta\epsilon_s' = (5.00W + 1)W^2 - 1, \quad \epsilon_s'' = 0.666W(5.00W + 1)^{1.5},$$  \hspace{1cm} (20)

$$a = 1/3 \quad \Delta\epsilon_s' = 10.3W + 35.7W^2 + 41.1W^3, \quad \epsilon_s'' = 0.495W + 3.42W^2 + 5.89W^3.$$  \hspace{1cm} (21)

The graphs of these formulae are shown in Figures 1-4. The empirical curves at 1 GHz have the form

$$\Delta\epsilon_s' = (0.1W + 0.8W^2)\epsilon_{w}'$$  \hspace{1cm} (26)

$$\Delta\epsilon_s'' = (0.1W + 0.9W^2)\epsilon_{w}''$$  \hspace{1cm} (27)

based on experimental results (Tiuri and others, 1984), are also shown.

**STRUCTURE-DEPENDENT MODELS**

Different dielectric mixing models will be presented that take into account the microscopic structure of the mixture. Ice and water are considered as the inclusion phases, the host material being air. The models take into account the shape of the inclusion particles. The mixing theories have been reported by Taylor (1965), Tinga and others (1973), and Polder and van Santen (1946).
The theory of Taylor leads to different formulae depending on the shape of the inclusions. For randomly oriented needles the formula is

\[
\varepsilon_m = -\frac{1}{2} d + \sqrt{\frac{1}{4} d^2 + \varepsilon_i + \frac{1}{3} \varepsilon_i (\varepsilon_i - 1)}
\]

where \( d = (\varepsilon_i - 1) (1 - \frac{5}{3} \phi) \). (28)

For randomly oriented disc-shaped inclusions the complex permittivity is

\[
\varepsilon_m = \frac{1 - \frac{2}{3} \phi (1 - \varepsilon_i)}{1 + \frac{\phi}{3 \varepsilon_i} (1 - 3 \phi)}
\]

For spherical inclusions the mixing formula is

\[
\varepsilon_m = -\frac{1}{4} b + \sqrt{\frac{1}{16} b^2 + \frac{1}{2} \varepsilon_i}
\]

where \( b = \varepsilon_i - 2 + 3 \phi - 3 \varepsilon_i \). This formula has also been derived by Böttcher (1952). Essentially the same results are obtained from the theory of Polder and van Santen.

For randomly oriented inclusions that are not discs, needles, or spheres but ellipsoids with known semi-axes the mixing formula is

\[
\varepsilon_m = 1 + \frac{1}{3} \phi (\varepsilon_i - 1) \sum_{a=1}^{3} \frac{\varepsilon_m (1 - N_a) + \varepsilon_i N_a}{\varepsilon_m (1 - N_a) + \varepsilon_i N_a}
\]

where \( N_1, N_2, \) and \( N_3 \) are the depolarization factors of the ellipsoids with semi-axes \( (a_1, a_2, a_3) \):

\[
N_j = \int_0^\infty \frac{a_1 a_2 a_3 dt}{2 \left( \frac{a_j}{a_j^2} \right)^{3/2} \left( \frac{a_1^2 + a_2^2 + a_3^2}{a_j^2} \right)^{1/2}}
\]

The equation

\[
N_1 + N_2 + N_3 = 1
\]

also holds. If two semi-axes of the ellipsoids are equal, two of the depolarization factors are also equal

\[
N_1 = N_2 = (1 - N_3)/2
\]

\[
= \frac{1}{2(1 - \varepsilon^2)} \left[ 1 + \frac{\varepsilon^2 \ln \left( \frac{1 - \sqrt{1 - \varepsilon^2}}{1 + \sqrt{1 - \varepsilon^2}} \right)}{2 \sqrt{1 - \varepsilon^2}} \right]
\]

for prolate ellipsoids, and

\[
N_1 = N_2 = (1 - N_3)/2
\]

\[
= \frac{e}{2(1 - \varepsilon^2)} \left[ \frac{1}{\sqrt{1 - \varepsilon^2}} \arccot \frac{e}{\sqrt{1 - \varepsilon^2} - \varepsilon} \right]
\]

for oblate ellipsoids. The ellipticity \( e \) is the ratio of the smaller semi-axis to the greater one. For ellipsoids of revolution in general the following mixing formula is valid:

\[
\varepsilon_m = 1 + \frac{1}{3} \phi (\varepsilon_i - 1) \times \left[ \frac{\varepsilon_m}{\varepsilon_i - 2N(\varepsilon_i - \varepsilon_m)} + \frac{2\varepsilon_m}{\varepsilon_m + N(\varepsilon_i - \varepsilon_m)} \right]
\]

where \( N = N_1 = N_3 \), the depolarization factor of the ellipsoid in the direction of the two equal semi-axes. For discs \( N = 0 \), for spheres \( N = 1/3 \), for needles \( N = 1/2 \), for prolate ellipsoids \( 1/3 < N < 1/2 \), and for oblate ellipsoids \( 0 < N < 1/3 \).

**Dry snow**

Formulae (28), (29), and (30) give the complex dielectric constant of dry snow. For randomly oriented needle-shaped ice inclusions

\[
e_d' = -\left( 1.075 - 1.792 \phi \right) +
\]

\[+ \sqrt{(1.075 - 1.792 \phi)^2 + 3.15 + 2.258 \phi} \]

\[
e_d' = \frac{1}{2} + 0.8333\phi +
\]

\[+ \frac{1.038 - 0.908 \phi + 1.493 \phi^2}{\sqrt{4.306 - 1.595 \phi + 3.210 \phi^2}} \]

For randomly oriented disc-shaped ice inclusions

\[
e_d' = \frac{0.700 \phi - 0.1035 \phi^2}{1 - 0.228 \phi} \]

For spherical ice inclusions

\[
e_d' = -\left( 0.288 - 1.612 \phi \right) +
\]

\[+ \sqrt{(0.288 - 1.612 \phi)^2 + 1.575} \]

\[
e_d' = \frac{1}{4} (3 \phi - 1) +
\]

\[+ \frac{1.2875 - 2.48 \phi + 4.84 \phi^2}{\sqrt{26.5 - 14.84 \phi + 41.6 \phi^2}} \]

The dielectric constant of ice is assumed to be \( \varepsilon_{\text{ice}} = 3.15 \) and \( \varepsilon_{\text{ice}}^2 \approx 3.5 \), and the assumption \( \varepsilon_{\text{ice}}^2 \ll \varepsilon_{\text{ice}} \) is made. The volume part of ice is \( \phi = \rho_i / \rho_{\text{ice}} \), \( \rho_i = 0.917 \text{ Mg/m}^3 \). In Figures 5 and 6 the results are illustrated together with the empirical results by Nyfors (1982).

**Fig. 5.** The real part of the dielectric constant of dry snow according to structure-dependent models. The points are experimental from Nyfors (1982).
Fig. 6. The loss tangent of dry snow according to structure-dependent models. The experimental points are from Nyfors (1982). The experimental points contain five sets of measurements. The data are relative to the densest sample in each set, which has been forced to fit the curve for needles.

Wet snow

Formulae (28), (29), and (30) give for the excess permittivity of snow due to liquid water at 1 GHz and 0°C:

Taylor, needles

$$\Delta \varepsilon'_w = 72.5W - 44.5 +$$

$$+ \sqrt{1980 - 3756W + 5265W^2}, \tag{43}$$

$$\frac{\varepsilon'_w}{\varepsilon'_w} = 0.833W - 0.5 +$$

$$\frac{22.25 - 43.33W + 60.4W^2}{\sqrt{1980 - 3756W + 5265W^2}}. \tag{44}$$

Taylor, discs

$$\Delta \varepsilon'_w = \frac{58.33W}{1 - 0.3295W} \tag{45}$$

$$\frac{\varepsilon'_w}{\varepsilon'_w} = \frac{0.6667W - 0.2172W^2}{1 - 0.6591W + 0.1086W^2}. \tag{46}$$

Taylor, spheres

$$\Delta \varepsilon'_w = 65.3W - 22.5 + \sqrt{506 - 2806W + 4258W^2}, \tag{47}$$

$$\frac{\varepsilon'_w}{\varepsilon'_w} = 0.75W - 0.25 + \frac{5.63 - 32.4W + 48.9W^2}{\sqrt{506 - 2806W + 4258W^2}}. \tag{48}$$

These formulae are depicted in Figures 7–10 together with the experimental results (Tiuri and others, 1984).

Fig. 7. The increase in the real part of the permittivity of snow due to liquid water according to structure-dependent models. N - Taylor, needles; D - Taylor, discs; S - Taylor, spheres; TINGA - Tinga and others, spheres. The broken line is empirical (Tiuri and others, 1984).

Fig. 8. The increase in the real part of the permittivity of snow due to liquid water (structure-dependent models). N - Taylor, needles; D - Taylor, discs; S - Taylor, spheres; TINGA - Tinga and others, spheres.

Fig. 9. (The increase in) the imaginary part of the permittivity of snow due to liquid water (structure-dependent models). N - Taylor, needles; D - Taylor, discs; S - Taylor, spheres; TINGA - Tinga and others, spheres. The broken line is empirical (Tiuri and others, 1984).
Fig. 10. The increase in the imaginary part of the permittivity of snow due to liquid water (structure-dependent models). N - Taylor, needles; D - Taylor, discs; S - Taylor, spheres; TINGA - Tinga and others, spheres.

Fig. 11. The increase in the real part of the permittivity of snow due to liquid water. Taylor model, prolate ellipsoids. The parameter N is the depolarization factor of the ellipsoids in the direction of the shorter semi-axes.

Fig. 12. The increase in the real part of the permittivity of snow due to liquid water. Taylor model, oblate ellipsoids. The parameter N is the depolarization factor of the ellipsoids in the direction of the longer semi-axes.

Fig. 13. (The increase in) the imaginary part of the Taylor model, prolate ellipsoids. The parameter N is the depolarization factor of the ellipsoids in the direction of the shorter semi-axes.

Fig. 14. (The increase in) the imaginary part of the Taylor model, oblate ellipsoids. The parameter N is the depolarization factor of the ellipsoids in the direction of the longer semi-axes.

Tinga and others

The theory of Tinga and others (1973) yields for spheres

\[
\epsilon_m^r = \frac{2 + \epsilon_1 + 2\phi(\epsilon_1 - 1)}{2 + \epsilon_1 - \phi(\epsilon_1 - 1)} .
\]  

(49)

For dry snow (ice-air mixture) assuming that \( \epsilon_{\text{ice}} = 3.15 - \text{j}\epsilon_{\text{ice}} \) and \( \epsilon_{\text{ice}}^r \ll \epsilon_{\text{ice}}^i \)

\[
\epsilon_5^r = \frac{1 + 0.835\phi}{1 - 0.417\phi} .
\]  

(50)

\[
\epsilon_{\text{ice}}^r = \frac{9\phi}{26.5 - 22.1\phi + 4.62\phi^2} .
\]  

(51)

These functions are given in Figures 5 and 6.

The excess permittivity of wet snow due to liquid water at 1 GHz and 0°C is

\[
\Delta\epsilon_5^r = \frac{261W}{90 - 87W} .
\]  

(52)

\[
\epsilon_5^r/\epsilon_w^r = \frac{9W}{8100 - 15660W + 7569W^2} .
\]  

(53)
These formulae are depicted in Figures 7–10 together with the experimental results (Tiuri and others, 1984).

**CHALOUPKA'S $Y$-FUNCTION**

Chaloupka, Ostwald, and Schiek have studied the dielectric properties of wet materials and the effect of the geometrical shape of the water particles (Chaloupka and others, 1980; Ostwald and others, 1980). They apply the mixing formula of Polder and van Santen for which they need to know the distribution function of the water inclusions. They calculate this function via the measurable real and imaginary parts of the dielectric constant of the mixture. They define the quotient $Y$ as

$$Y = \frac{\varepsilon_m' - \varepsilon_{dry}}{\varepsilon_m'}$$

where $\varepsilon_{dry}$ is the permittivity of the dry material, and $\varepsilon_m'$ the complex permittivity of the mixture of this dry material and water.

The central feature in this $Y$-function from the point of view of tentative mixing theories, is the assertion stated by Chaloupka, Ostwald, and Schiek that the $Y$-function is independent of the water content. In other words, any amount of liquid water augments in the same rate both the real and imaginary part of the material. Applied to the air–water mixture discussed in this paper, the $Y$-function will be

$$Y = \frac{\Delta \varepsilon}{\varepsilon_s}.$$  

\[(55)\]

Therefore the flatness of a plot of the $Y$-function versus water volume $W$ is a measure of the pertinence of the Chaloupka approach to a water–mixture model.

If the $Y$-function is flat for a mixing model, then its reciprocal $1/Y$ will also be flat. It seems more natural to compare the change in the imaginary part to the change in the real part due to liquid water, i.e. to calculate the function $1/Y = \varepsilon_s' / \Delta \varepsilon$. Experimental results (26) and (27) give support to the thesis of Chaloupka and others, namely that $1/Y = \tan \theta_W$. Figures 15 and 16 show the behaviour of the $Y$-function according to mixing models discussed in this paper.
in the model is to assume a distribution function for the axial ratio of the water inclusions. Nevertheless, the theoretical treatment will become tedious, and the formulae resulting from the exponential models may be more suitable for engineering applications.

Finally, Figures 15-16 show that Chaloupka's sense the exponential models are the better the nearer the parameter  is to unity. From the structure-dependent models, the needle and disc models of Taylor are very good, but the sphere models of Taylor and especially Tinga and others are extremely poor.

REFERENCES


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