MOTION OF SUB-FREEZING ICE PAST PARTICLES, WITH APPLICATIONS TO WIRE REGELATION AND FROZEN SOILS

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ABSTRACT. Existence of a very thin layer of adsorbed water adjacent to particles embedded in ice allows relative motion between ice and particles even at sub-freezing temperatures if there are either applied stresses or macroscopic temperature gradients. Theoretical analysis of such motion involving a single sphere demonstrates that such motion is dominantly due either to "viscous" deformation in the ice or to mass transport in the liquid layer at temperatures below the nominal pressure melting-point, depending on the ratio of the sphere's radius to a temperature-dependent "transition radius." This result should also hold for motion of a cylinder (for which the creeping flow problem has no known analytical solution). Reviewing data on wire regelation at sub-freezing temperatures in the context of this analysis suggests that all "anomalous" data correspond to cases in which wire radii were greater than the transition radius, leading to dominance of ice-deformation effects. Ice motion past very small particles, on the other hand, is essential accommodated entirely by mass transfer through the liquid layer. This result lends support to the "rigid-ice" model of frost heaving as proposed by R.D. Miller and co-workers, and permits approximate analysis of ice movement through a porous soil. In all cases involving relative motion between ice and particles at sub-freezing temperatures, the existence of macroscopic temperature gradients plays an important role.


1. INTRODUCTION

The long-standing notion that glaciers cannot slide over their beds at sub-freezing temperatures (i.e., at temperatures below the pressure melting-point) has recently been challenged. Shreve (1984) modified the Nye-Kamb version of glacier-sliding theory (Nye, 1969, 1970; Kamb, 1970) to account for the existence of a liquid-like layer at the ice-rock interface at sub-freezing temperatures, and showed that sliding rates, albeit very small, could lead to substantial total displacement over a period of many years. Field studies by Echelmeyer and Zhongxiang (in press) and by Hallet and others (in press) provide evidence of glacier sliding where basal ice is at least locally at sub-freezing temperatures, although the exact mechanisms of sliding in these cases remain poorly understood.

However, the concept of ice "sliding" past obstacles at sub-freezing temperatures is not new in the literature on frozen soils, where it has been promoted since at least 1972 by R.D. Miller and co-workers. The most common mode of ice-lens formation, known as "secondary heaving" (Miller, 1972), is thought by Miller and co-workers to involve actual motion of sub-freezing pore ice towards the ice lens, this motion being rendered possible by the existence of very thin films of "adsorbed" water at ice-mineral grain interfaces. The exposition of the concept of pore-ice motion as given by O'Neill and Miller (1985) is so lucid that it is well worth quoting them at some length. Considering first a single grain embedded in ice, "at temperatures somewhat below 0°C the grain ought to be surrounded by a film of unfrozen liquid in
equilibrium with the ice. If a temperature gradient is imposed, thermal equilibrium of water and ice at the interface is inconsistent with mechanical equilibrium in the hydrostatic field induced by surface adsorption forces. Whereas the thermal gradient induces asymmetry of film thickness, the action of adsorption forces is to center the grain within its liquid shell. Thus the temperature field constantly acts to diminish the film thickness on the cold side, while surface forces seek to retain that film thickness by removing ice from the warm side and transporting the resulting unfrozen water to the cold side, where it refreezes. The grain ought to migrate up the temperature gradient and its velocity should increase as it moves into an ever warmer environment with a corresponding increase in average thickness of the film, expediting the flow of liquid by which the centering tendency is expressed (O’Neill and Miller, 1985, p. 283).

The predicted migration of grains through ice was observed by Hoekstra and Miller (1967) and by Römkens and Miller (1973). As a corollary, O’Neill and Miller (1985, p. 283) proposed that:

“If individual grains migrate through stationary rigid ice, traveling up a temperature gradient, then rigid ice, that largely fills interstices between stationary grains ought to migrate down a temperature gradient. If the ice is inherently rigid, this movement is not flow but continuous regelation. Crystalline ice, everywhere bounded by liquid in both adsorption and capillary space, is continuously melting and reforming in a manner consistent with the geometry imposed by the array of soil grains.”

This scenario of "thermally induced regelation", which must be distinguished from the better-known phenomenon of pressure-induced regelation, is illustrated as well by Figure 1. It should be noted that the adsorbed liquid films are very much thinner (only a few nm at −1°C) than the liquid layer in the case of pressure-induced regelation (cf. Nye, 1967; Gilpin, 1979, 1980(c)). O’Neill and Miller’s description of the pore ice as "rigid" deserves further examination. By ‘rigid’, O’Neill and Miller meant that plastic deformation of the pore ice is of negligible importance in secondary heaving, and indeed they drew on rheological data to develop a semi-quantitative argument (p. 285) showing plastic deformation should be negligible for silty soils (grain-size c. a few tens of μm), which are known to be highly susceptible to frost heaving.

In problems of glaciological interest involving mixtures of mineral grains and sub-freezing ice — say, the mechanics of "cold-based" glaciers resting on permafrost — the assumption of "rigid" ice may fail. We may anticipate from Shreve's (1984) results — and will later demonstrate — that "thermally induced regelation" through a porous material with a typical grain-size of more than c. 100 μm will involve an important element of plastic deformation as well.

In section 2, we develop a theoretical model of ice flow past a single sphere at sub-freezing temperatures. This development draws upon the analysis of the motion of temperate ice past a sphere (Watts, unpublished) and Gilpin's (1979) model of the liquid layer at the interface between sub-freezing ice and an embedded particle. This study, besides complementing Shreve's (1984) modified sliding theory, provides insight into the results of some experiments on wire regelation and also lays the basis for an approximate analysis of ice movement through porous materials, presented in section 3.

2. ANALYSIS

Ice flow

The passage of a particle through sub-freezing ice is conceptually similar to the more familiar temperate-ice situation. Ice mass may be re-distributed either by plastic deformation or by a melting-refreezing process made possible by a thin, continuous liquid layer at the particle–ice interface. A major difference from the temperate-ice situation arises because there may be macroscopic temperature gradients within the ice. In the absence of forces applied to the particle, it is in fact such temperature gradients that cause particle motion. Furthermore, explicit analyses of flow in the liquid layer is required (cf. Shreve, 1984).

The first part of this analysis directly parallels that of Watts (unpublished, p. 25-27). Figure 2 shows the geometry to be considered. For convenience, the sphere of radius R is considered fixed at the origin of coordinates, the usual spherical coordinates r (radial distance), θ (polar angle), and ϕ (azimuthal angle) are used. The liquid-layer thickness, greatly exaggerated here, would actually be non-uniform.

*Rather than cite the numerous papers dealing with the apparent existence (or absence) of such a layer, we refer the reader to sources cited by Gilpin (1979, p. 236).
convention for subscripts). The ice is assumed to be an incompressible fluid with Newtonian viscosity \( \eta_i \). Inertial effects are so small that the equation of motion reduces to

\[
\eta_i \nabla^2 V = \nabla p
\]

which combined with the incompressibility condition

\[
\nabla \cdot V = 0
\]

implies that \( \nabla^2 p = 0 \). Symmetry conditions, along with the constraint that \( p \to 0 \) as \( r \to \infty \), lead to a solution for the pressure

\[
p = \frac{\eta_1 A \cos \theta}{r^2}
\]

where \( A \) is a constant to be determined. Equation (3) may now be substituted into Equation (1) to solve for \( V \), and standard relationships from fluid mechanics (e.g. Bird and others, 1960, p. 90) used to find the stresses. The aforementioned far-field condition on flow is equivalent to the boundary condition

\[
v_r = U \cos \theta, \quad v_\theta = -U \sin \theta, \quad \text{as} \quad r \to \infty.
\]

Furthermore, the liquid layer at the ice-sphere interface lubricates that boundary, hence

\[
\sigma_{rl} = 0, \quad \text{on} \quad r = R.
\]

Equation (5) is sufficiently accurate as long as

\[
h/R \ll 1 \quad \text{and} \quad \frac{1}{h} \frac{\partial h}{\partial \theta} \ll 0.1
\]

where \( h \) is the liquid-layer thickness. With these boundary conditions, the velocities and stresses are readily found to be

\begin{align}
  v_r &= (U + A \frac{A \cos \theta}{r^2}) \\
  v_\theta &= -(U + A \frac{A \sin \theta}{2r}) \\
  v_\phi &= 0 \\
  \sigma_{rr} &= -3\frac{\eta_1 A}{r^2} \cos \theta
\end{align}

with all other \( \sigma_{ij} \) vanishing.

If the ice were temperate, the constant \( A \) would be determined (cf. Nye, 1967; Watts, unpublished) by solving for the temperature distribution and imposing the constraint that the temperature \( T \) and normal stress \( \sigma_{rr} \) everywhere on \( r = R \) satisfy the pressure-melting relationship \( T = \sigma_{rr} \), where \( T \) is measured relative to 0°C. In the present case of sub-freezing ice, \( T \) and \( \sigma_{rr} \) are not so simply related. Determination of \( A \) requires that we solve not only for the temperature field but also for the liquid-layer thickness \( h \) everywhere at the sphere-ice interface, \( h \) and \( T \) being functionally related (cf. Gilpin, 1979, p. 239). We will find two expressions for \( h \) and fix the constant \( A \) by requiring that these two expressions be equivalent.

### Flow within the liquid layer

We will now adopt the model of the liquid layer at sub-freezing temperatures proposed by Gilpin (1979) and applied by him to problems of wire regelation, particle migration, and ice lensing (Gilpin, 1979, 1980[a], [b], [c]). Gilpin's fundamental assumption - one motivated by a variety of experimental phenomena - is that the chemical potential of water, but not of ice, is lowered in close proximity to a solid surface, the thermodynamic effect being given by (Gilpin, 1979, p. 238)

\[
\mu_w = \mu_{wb} - \mu_{w0}
\]

where the chemical potential \( \mu_{wb} \) of water is given by the difference between the chemical potential \( \mu_{wb} \) of bulk water and the change in chemical potential \( \mu_{w0} \) due to the surface. For mathematical convenience, Gilpin assumed \( \mu_{w0} \) to be given by

\[
\mu_{w0} = ay^{-\alpha}
\]

where \( a \) and \( \alpha \) are constants and \( y \) is the distance measured normal to the surface and towards the ice. Equation (9) is valid only for \( y > y_w \) where \( y_w \) is of the order of a few molecular dimensions" (Gilpin, 1979, p. 238).

The chemical attraction of water to the surface will be manifested in part by an increase in water pressure near the surface (Gilpin, 1979, p. 239; Equation (5)):

\[
\hat{P}_w = \frac{a}{y_w} y^{-\alpha}
\]

where \( y_w \) is the specific volume of water and \( P_w \) is the water pressure relative to some datum, here taken as the no-motion, no-temperature-gradient state. Equilibrium at the ice-water phase boundary leads to the additional condition (Gilpin, 1979, p. 239; Equation (11)):

\[
h \frac{\cos \theta}{2} \frac{\partial h}{\partial \theta} = -\frac{\partial \mu_{wb} \partial T}{\partial P} - \frac{h^2}{\eta_w \partial T}
\]

where \( \eta_w \) is the viscosity of water, \( h \) the liquid-layer thickness, \( y_w \) the water pressure relative to some datum, \( T \) absolute temperature (K), and \( \mu_{wb} \) surface tension of ice-water interface, \( K \) mean curvature of ice-water interface, \( L \) latent heat of fusion, \( T_a \) absolute temperature (K), and \( P_{wb} \) water pressure at ice-water interface.

Equation (11) shows that the "melting" temperature depends not only on pressure but also on liquid-layer thickness. Variations in liquid-layer thickness, hence water pressure, lead to flow along the liquid layer. Mass conservation requires that such flow be balanced by melting or freezing at the phase boundary. This conservation relationship may be readily put into mathematical form.

Let \( q_w(\theta) \) be the "upward" mass-flow rate through the liquid layer at the polar angle \( \theta \) (cf. Fig. 2). Assuming that the water behaves in a linearly viscous fashion and that Equations (6) are valid, we may write, in analogy to Gilpin (1979, p. 240; Equation (17)):

\[
q_w(\theta) = \frac{n_w \tau_w(\theta)}{6 \eta_w \partial T} \frac{dP_w}{d\theta}
\]

As long as the liquid-layer thickness is constant in time, \( q_w(\theta) \) must be balanced by ice influx \( q_i(\theta) \), given by the surface integral

\[
q_i(\theta) = \frac{1}{v_i} \int_{f \to n} q R dA
\]

where \( n \) is the unit outward normal from the phase boundary and the integral is taken over that part of the surface spanned by polar angles \( \theta \) to \( \eta \). Substituting Equation (7a) into Equation (13) and integrating, then equating the fluxes \( q_i(\theta) \) and \( q_w(\theta) \), we find after some rearrangement

\[
6n_w \int_{v_i}^{v_i} \hat{P}_w(\theta) = \frac{\frac{\partial \tau}{\partial T}}{n_w} \int_{v_i}^{v_i} \frac{dP_w}{d\theta}
\]
and using Equation (11) to eliminate \( P_{wh} \). For \( h \gg y_0 \), we find

\[
6\nu_w^2 \left( \frac{\partial P_{wh}}{\partial h} \right) R^2 \sin \theta = -\frac{Lh^2}{T_a} \frac{d\theta}{dh} + \frac{a_{ah}^2 - a_{ah} \nu^2}{\nu_w \Delta \nu} \frac{dh}{d\theta}
\]

(16)

where the contribution of the term \( dK/dB \) may be shown to be negligible because of the assumption that \( h \) varies little with \( B \) (Equation (6)).

Temperature-stress relationship at the phase boundary

We now derive the expression that replaces the pressure-melting condition used in Watts' (unpublished) analysis of a sphere moving through temperate ice. Equation (11) may be re-written as

\[
\alpha h = -\Delta \hat{P}_1 + \nu_w \alpha_{tw} - \frac{LT}{T_a} \frac{d\hat{P}}{d\theta}
\]

(17)

where \( \hat{P}_1 \), the "pressure" in the ice normal to the phase boundary, is measured relative to the no-motion, no-temperature-gradient state. When Equation (6) holds,

\[
a_{ah}^2 \frac{dh}{d\theta} = \left( \frac{6\nu_w^2}{\nu} \right) \Delta R^2 \sin \theta = -\frac{Lh^2}{T_a} \left( \frac{R}{2k_1 + k_s} \right) \left[ \frac{3k_1 G_T + \frac{L}{\nu} \left( \frac{U + A}{R} \right)}{v_1} \right] \sin \theta
\]

(20)

Equations (16) and (18) constitute two equations for the three unknowns \( h, T, \) and \( A \). We now use energy-conservation considerations to find a third expression.

Temperatures at the ice-sphere interface

The temperature distribution at the ice-sphere interface is readily found by adapting the analysis of Gilpin (1979, p. 241-42) to the problem at hand. The essential difference between our formulation and Gilpin's is that he implicitly used the rigid-ice model; with the present model the normal component of ice velocity at the interface is of magnitude \( U \cos \theta \) instead of simply \( U \cos \theta \). The temperature is therefore (cf. Gilpin, 1979, p. 242)

\[
\hat{T} = \hat{T}_u + \frac{R}{(2k_1 + k_s)} \left[ \frac{3k_1 G_T + \frac{L}{\nu} \left( \frac{U + A}{R} \right)}{v_1} \right] \sin \theta
\]

(19)

where \( \hat{T}_u \) is "undisturbed" temperature at center of sphere, \( k_1, k_s \) are thermal conductivities of ice and sphere, respectively, and \( G_T \) is imposed temperature gradient \( \partial T/\partial z \) in ice far from the sphere (cf. Fig. 2). Equation (19) is valid if convected heat, as well as work done by the sphere's motion, are of negligible importance in comparison to conduction, as is almost always true (Philip, 1980, p. 195-98), and if the temperature drop across the liquid layer is negligible. This latter condition requires (Nye, 1967) that \( h/R \ll k_w/k_s \), where \( k_w \) is the thermal conductivity of water; this condition is nearly always met as long as we restrict our attention, and justifiably so, to geological materials. A further assumption in deriving Equation (19) is that there is no net heat flow either towards or away from the sphere. This last assumption requires some further discussion.

Drake and Shreve (1973, p. 66) found in their wire-regelation experiments with ice at 0°C that, at sufficiently large driving forces, the wires left behind themselves a trace of water and vapor, the volume of which could be explained only if there had been a net flow of heat toward the wires. Formation of the trace was associated with a transition from a "slow" to a "fast" mode of wire motion. A similar sort of transition was observed in wire-regelation experiments using ice at temperatures below 0°C (Telford and Turner, 1963; Gilpin, 1980[c]), although these authors did not discuss any water trace behind the wires. The exact reasons for the transition remain uncertain (Drake and Shreve, 1973, p. 69; Gilpin, 1980[c], p. 446-47). It seems plausible that such a transition might also occur for a sphere moving through ice, although there are no experimental data to test this hypothesis properly. Our analysis should therefore be considered restricted to the "slow" mode of motion.

We now substitute Equation (19) into Equations (16) and (18) to eliminate \( T \), finding after some re-arrangement

\[
\frac{\partial h}{\partial \beta} = -\frac{\partial \hat{P}_1}{\partial R} \frac{d\beta}{d\theta} + \frac{Lh^2}{T_a} \left[ \frac{3k_1 G_T + \frac{L}{\nu} \left( \frac{U + A}{R} \right)}{v_1} \right] \sin \theta
\]

(21)

Liquid-layer thickness

Before proceeding to the solution of Equations (20) and (21), it is useful to recast these equations into dimensionless form. Following Gilpin (1979, p. 292), the liquid-layer thickness will be re-expressed as

\[
h = h_c (1 + h')
\]

(22)

where \( h_c \) is the equilibrium thickness for a stationary sphere, given by

\[
a_{ah}^2 - a_{ah} \nu^2 = \frac{L - \hat{P}_u}{T_a} + \nu_w \alpha_{tw} K
\]

(23)

and the dimensionless deviation \( h' \) from this thickness must be much less than one, as implied by Equation (6b). The characteristic temperature will be taken as (Gilpin, 1979, p. 242)

\[
\hat{T}_C = \hat{T}_u - \frac{h}{L} \nu_w \alpha_{tw} K
\]

(24)

We may now write three characteristic velocities that arise out of Equations (20) and (21), two of them identical to those defined by Gilpin (1979, p. 242), viz.:
Equation (32) is then our fundamental relationship expressing the ice-flow rate (or equivalently, the sphere's velocity) as a function of the macroscopic temperature gradient, the applied force (or equivalently drag), and material parameters.

Relative efficacy of regulation and viscous deformation: the "transition radius"

Pressure-induced flow. It is useful to consider the special case \( G_T = 0 \) because it most clearly illustrates the way in which the sphere's rate of motion depends on its size. When \( G_T = 0 \), Equation (32) becomes, after some re-arrangement:

\[
P_d = 8\pi U \left( \frac{R^2}{2R^2 + R_s^2} \right).
\]

For a given \( U \), the drag is a maximum when \( R = R_s \), where

\[
R_s = \left( \frac{k_1}{v_w} \right) \left( \frac{n_{inj}}{n_{inj}} \right)^{1/3} \frac{R_e}{R_c}.
\]

For \( R \ll R_s \), the motion of the sphere is accommodated primarily by regulation at sub-freezing temperatures and flow through the liquid layer (which, we emphasize, is only a few nm thick); for \( R \gg R_s \), the regelative process is very inefficient and the sphere's motion is accommodated primarily by deformation of the ice. At \( R = R_s \), neither the regelative nor the deformational process is particularly efficient and the resistance to motion is the greatest.

It should be noted that, not surprisingly, our expression (Equation (34)) for the transition radius (with \( G_T = 0 \)) is identical, to within a small numerical constant, to the "transition wavelength" of Shreve's (1984, p. 343; Equation (11)) sliding theory for cold-based glaciers. An analogous situation exists with respect to the Nye-Kamb glacier-theory and Watts' (unpublished) analysis for sphere motion through temperate ice.

Flow with macroscopic temperature gradient. The essential physics of the sphere migration are not altered by the existence of a non-zero value of \( G_T \), although the transition radius is altered. Equation (32) may be re-written in dimensional form as

\[
P_d = 8n_{inj} U \left( \frac{R^2}{2R^2 + R_s^2} \right).
\]

where all symbols, including \( R_s \), are as defined previously. The transition radius, now denoted by \( R_{\text{trans}} \), is still defined by the condition \( \partial P_d / \partial R = 0 \). This leads to a quartic equation for \( R_{\text{trans}} \):

\[
R_{\text{trans}}^2 - \left( \frac{k_1}{v_{inj} T_a} \right) \left( \frac{L}{v_{inj} T_a} \right) G_T R_e^2 + \left( \frac{k_1}{2k_1 + k_2} \right) \left( \frac{L}{v_{inj} T_a} \right) G_T R_s^2 = 0.
\]

Explicit evaluation of the quartic is extremely tedious and unnecessary, as we may easily place bounds on the value of \( R_{\text{trans}} \). Only one positive, finite value of \( R_{\text{trans}} \) can satisfy Equation (36). Clearly, if \( G_T = 0 \), we must find \( R_{\text{trans}} = R_s \). Furthermore, if one examines the functional behavior of \( \partial R_{\text{trans}} / \partial G_T \) and \( \partial R_{\text{trans}} / \partial G_T \), it is easy to show that for \( U > 0 \), the extreme value of \( R_{\text{trans}} \) is \( 4^{1/3} R_s \), and that such an extremum is a minimum. (This extremum in fact occurs as \( G_T = \) -). Hence, 4\( \cdot R_s \) is
In Figure 3, we show the upper bound on \( R_{\text{trans}} \) as a function of \((-T)\), the temperature below 0°C. The temperature dependence of \( R_{\text{trans}} \) is contained implicitly (cf. Equation \( 37 \)) in the temperature dependence of ice viscosity \( n_i \). The characteristic liquid-layer thickness \( \theta \) is assumed to be given by \( (\text{Gilpin, 1980}^{[c]}) \).

\[ h_e = 3.5 \text{ nm K}^{1/2} \text{ P}^{1/4} \]

This expression for \( h_e \) arises from Gilpin's best fit of his theoretical predictions to his data on wire regulation, discussed next.

**Transition radius: relevance for wire-regulation studies**

It is difficult to compare the theoretical predictions above to experimental data, because the only set of experiments on motion of a sphere through ice (Townsend and Vickery, 1967) were conducted with the ice at 0°C and atmospheric pressure. Our predictions of transition radius for motion of sub-freezing ice past a sphere may, however, provide guidance for interpreting results of "wire-regulation" experiments (e.g., Telford and Turner, 1965). Force-balance considerations analogous to those presented in the next section (cf. Philip, 1980, p. 203) indicate that \( P_d / P_{\text{OB}} \) should be of order \( R/L_f \), where \( L_f \) is the thickness of the zone through which ice moves. \( R/L_f \) is necessarily less than one. We therefore conclude that typical values of \( P_d \) of interest in frozen-ground phenomena are usually no more than a few bars.

FIG. 3. Transition radius between regulation-dominated and creep-dominated flow, for various driving stresses \( P_d \): Effective viscosity of ice is dependent on both temperature and \( P_d \).

\[ R_{\text{trans}} \approx R_s, \text{ which is a rather narrow range (4·1}/3 = 0.62). \]

The transition radius and the "rigid-ice" approximation for frozen soils

The transition radius for \( R_{\text{trans}} \) illustrated in Figure 3 are also directly relevant to the question raised in the introduction about the correctness of the rigid-ice formulation in the theory of frost-heaving (O'Neill and Miller, 1985). Although a soil is composed of a multitude of grains, with none likely to be spherical, the results of our analysis of ice flow past a single sphere should be illustrative of the basic physics involved in ice movement through a soil.

Physically plausible conditions under which substantial amounts of ground-freezing and heave occur almost never involve overburden loads of more than a few bars (cf. O'Neill and Miller, 1985). Force-balance considerations analogous to those presented in the next section (cf. Philip, 1980, p. 203) indicate that \( P_d / P_{\text{OB}} \) should be of order \( R/L_f \), where \( L_f \) is the thickness of the zone through which ice moves. \( R/L_f \) is necessarily less than one. We therefore conclude that typical values of \( P_d \) of interest in frozen-ground phenomena are usually no more than a few bars.

Figure 3 then shows that the transition size \( R_{\text{trans}} \) should be no less than c. 100 \( \mu m \), and at least several times that value at temperatures above about -2°C. Because soils that exhibit significant heave are characterized by grain sizes of less than several tens of microns, it seems clear that viscous deformation of the ice should play a negligible role in the overall ice motion. O'Neill and Miller's (1985) treatment of the pore ice as "rigid" therefore seems quite reasonable. Neglect of viscous deformation also permits us to construct an approximate theory for pore-ice motion.
because Philip (1980) has presented results for ice motion past arrays of cylinders for the pressure-melting regime. We may directly adapt his analysis—in much the same way as we adapted Watts' (unpublished) work in the preceding section—to solve the problem of thermally induced regulation of ice through an array of cylinders, which we will take as our highly idealized model of a soil.*

The basic result we adapt from Philip's (1980) analysis is his expression for the temperature field at the ice-cylinder interface. Obviously, when there is a macroscopic temperature gradient \( G_T \neq 0 \), the mean temperature of individual cylinders may differ, but the results for variation of temperature about any cylinder remain valid. We then proceed as in Gilpin's (1979) analysis to solve for the liquid-layer thickness and, finally, for the drag force.

For an infinite square array of cylinders, each of radius \( R \) and with array spacing \( H \) (Fig. 4), and assuming that the thermal conductivity of the cylinder \( k_i \) equals that of the ice \( k_i \), the temperature field at any point may be represented mathematically by an infinite series, each term representing the contribution of an individual cylinder which behaves formally as a "thermal dipole." The series may formally be summed by contour-integral methods but the solution would be in terms of elliptic functions (Philip, 1980). Philip (1980) has presented what he terms "a good approximation" for the temperature field in terms of simpler functions. We have modified his solution to include the case of \( G_T \neq 0 \) (cf. Gilpin, 1979, p. 242) and changed to a cylindrical coordinate system, where the origin may be taken at the center line of any cylinder of interest so long as we look only at temperature variations instead of absolute values.

The temperature variation at the cylinder-ice interface may be expressed as

\[
\nabla T = \\frac{2G_Tk_i + \rho_iLU}{2k_i}\left[ \frac{\sinh \left( \frac{2\pi}{H} R \cos \theta \right)}{H} - \frac{\cosh \left( \frac{2\pi}{H} R \cos \theta \right)}{H} \right] + \frac{2R \cos \theta}{H} \left[ \frac{\sinh \left( \frac{2\pi}{H} (H - R \cos \theta) \right)}{H} - \frac{\cosh \left( \frac{2\pi}{H} (H - R \cos \theta) \right)}{H} \right]
\]

where \( \rho_i = 1/\nu_i \) is the density of ice. Later calculations (e.g. of the drag force) would essentially involve integrating this expression, a task that appears quite daunting. One may obtain useful approximations to Equation (37) by expanding in terms of trigonometric functions. The larger the quantity \( R/H \), that is, the closer the cylinders are to each other, the more terms must be kept to yield a sufficiently close approximation. For the problem at hand, we may elucidate the basic physics even with a fairly "low order" expression. This restricts us to fairly large cylinder spacings but makes the mathematical manipulations much easier.

The lowest-order expansion of Equation (37) that still includes the effect of multiple cylinders may be shown to be

\[
\nabla T = \frac{2G_Tk_i + \rho_iLU}{2k_i}\left[ \left( 1 - \frac{4\pi^2}{\sin^2 \theta} \right) \cos \theta + \left( \frac{3}{\pi} - \frac{2\pi^2}{\sin^2 \theta} \right) \cos^2 \theta + \left( \frac{16}{\pi} - \frac{2\pi^2}{\sin^2 \theta} \right) \sin^2 \theta \cos \theta \right] \quad (38)
\]

where \( \beta = (\pi R/H)^2 \). This may be shown to be a very good approximation as long as \( H/2R \) (i.e. cylinder spacing/cylinder diameter) is greater than about 2.87 (whereas a cubic close-packed array would have \( H/2R = 1 \)).

Equation (38) is combined with an expression for the variation in liquid-layer thickness (cf. Gilpin, 1979, p. 241-42, and our Equation (16)):

\[
12\pi^4k_i\frac{\partial \theta}{\partial T} \sin \theta = -\frac{Lh^4}{\alpha T_d} \frac{\partial \theta}{\partial h} + \frac{\alpha h^2 - c_T \nu_i}{\nu_i \alpha} \frac{\partial \nu_i}{\partial h} \quad (39)
\]

The procedure for calculating the drag proceeds exactly as in Gilpin (1979) and in the foregoing analysis, hence it need not be repeated here. We will again restrict our attention to cases in which the liquid-layer thickness does not vary greatly around the cylinder, i.e. the dimensionless thickness perturbation \( h' \ll 1 \). The drag force per unit cross-sectional area of the cylinder, \( P_d \), must now be understood as \( P_d = F/2IR \), where \( F/\ell \) is the drag force per unit length of the cylinder. The relationship between \( U \), \( P_d \), and \( G_T \) may be written as

*Philip stated no restrictions on values of \( R/H \) for which this expression holds.

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* An array of spheres would seem to be a much better idealization but formal analysis for such an idealized geometry appears intractable. A "self-consistent" approximation such as used in some analyses of the properties of composites and cracked solids (e.g. O'Connell and Budiansky, 1974) might be useful in future work.

† For a linear rather than square array, the contour-integral method leads to solutions in terms of elementary functions. We have used this method to verify Philip's results for the special case \( k_i = k_s \).
In the absence of externally applied forces, the existence of a non-zero drag \( P_D \) for each cylinder in an infinite array requires a macroscopic gradient of ice pressure. This becomes clear from simple force-balance considerations. The drag per unit length \( F/\ell \) must be balanced by a gradient in ice pressure \( p_1 \) such that

\[
F/\ell = - \left( \int_{x_0}^{x_0+H} p_1 (x, x_0 + H/2) - p_1 (x, x_0 - H/2) \, dx \right)
\]

(41)

where \( x_0 \) is the z-coordinate of the "row" of cylinders of interest and \( x_0 \) is an arbitrary point along such a row. The mean gradient of ice pressure \( G_p \) is conveniently defined as (cf. Philip, 1980, p. 203)

\[
G_p = \left( \int_{x_0}^{x_0+H} p_1 (x, x_0 + H/2) - p_1 (x, x_0 - H/2) \, dx \right) / H.
\]

(42)

Combining Equations (41) and (42), and our definition of \( P_D \) for a cylinder, gives \( G_p = -2RP_d/H^2 \). Equation (40) may therefore be written as

\[
U = \left( \frac{1}{\delta n_{w,v}} \right) (v_{1, w})^2 \left( \frac{\eta_{c, w}}{R} \right) \left( \frac{P_D}{R} \right) \left( \frac{L G_p}{v_{1, w}} \right) \left( \frac{1 + \frac{1}{2} \frac{4e^{-2N}}{\pi (B + B^2)} \right)
\]

(40)

which is identical to the thermodynamic condition for cessation of flow in the liquid layer (Gilpin, 1980[a], p. 919).

It should be noted from Equation (43) that steady flow necessarily requires \( U \) to be uniform throughout the space-filling array of cylinders. (Recall that we have followed O'Neil and Miller (1985) in assuming that the pore ice forms a connected network.) Because the liquid-layer thickness \( \eta_{c, w} \) varies with temperature, hence position, steady flow is only possible if the gradient of \( \phi \) (Equation (44)) varies spatially in such a way as to make \( U \) uniform. B. Hallet (personal communication) has suggested that the requirement of spatially uniform \( U \) holds only if "through-flow" of \( H_2O \) is restricted to the solid ice, and that in the pore space of a real frozen soil, with unfrozen "capillary" water in addition to adsorbed water films, steady flow of \( H_2O \) could occur even if \( U \) varied spatially. K. Hutter (personal communication) has also pointed out that \( U \) need not be spatially uniform if particle spacings are not everywhere the same. However, without prescribing \( U \) in some fashion, it is difficult to see how an analysis of the pore-ice regulation process could have been developed. It is unlikely that the conclusion of spatially uniform \( U \) is grossly in error.

We propose to define an apparent hydraulic conductivity \( K_R \) on the basis of Equation (43). This apparent hydraulic conductivity is defined by a Darcy's law type of expression

\[
\left( \frac{v_{1, w}}{g} \right) U = - \frac{\phi K_R}{\eta_{w,v}} \frac{d\phi}{dz}
\]

(47)

where the factor of \( v_{1, w}/g \) on the left-hand side "corrects" for the density difference between water and ice. For the geometry considered,

\[
K_R = \left( \frac{1}{12} \right) \left( \frac{\eta_{w,v}}{v_{1, w}} \right) \left( \frac{\eta_{c, w}}{R} \right) \left( \frac{H}{R} \right) \left( \frac{1}{\phi} \right)
\]

(48)

or relating \( H/R \) to porosity \( \phi \) by straightforward geometrical considerations,

\[
K_R = \left( \frac{\eta_{w,v}}{v_{1, w}} \right) \left( \frac{\eta_{c, w}}{R} \right) \left( \frac{H}{R} \right) \left( \frac{1}{\phi} \right) \left( \frac{1}{1 - \phi} \right)
\]

(49)

For more general, non-cubical packings of cylinders, a reasonable functional form for \( K_R \) may be

\[
K_R = \gamma \left( \frac{\eta_{w,v}}{v_{1, w}} \right) \left( \frac{\eta_{c, w}}{R} \right) \left( \frac{H}{R} \right) \left( \frac{1}{\phi} \right) \left( \frac{1}{1 - \phi} \right)
\]

(50)

where \( \gamma \) is a numerical constant of \( O(1) \) and the function \( f \) depends on porosity and the geometry of packing.

The apparent hydraulic conductivity defined by Equation (48) is shown as a function of temperature in Figure 5 for several choices of \( R \). The array was assumed to be in cubic close packing, \( H/R = 2 \). The hydraulic conductivity values in Figure 5 are rather low in comparison with most measured values for frozen soils (e.g. Burt and Williams, 1976; Horiguchi and Miller, 1980, 1983). This result is actually not particularly surprising. In the present
model, all of the pore space—aside from the extremely thin unfrozen films of water at ice–grain interfaces—is assumed to be ice-filled. In a real porous medium, the great variability in pore sizes and shapes leads to a continuous variation in ice content as a function of sub-freezing temperature. The rates of wire movement are so exceedingly small at temperatures much below 0°C (Gilpin, 1980[c]) that experimenters might be tempted to use large loads to increase wire speeds. Our results make it clear that wire radii would have to be quite small, perhaps c. 10 μm, to avoid appreciable deformation in the ice. This may lead to restrictions on the types of wires used.

In detail, the various modes of \( H_2O \) mass transfer will probably interact. Furthermore, the present analysis, when considered in the context of the work by Gilpin (1980[a]) and O'Neill and Miller (1985), suggests that the gradient of the generalized potential \( \nabla = \mathbf{F}_t + \frac{(LT^2)_t}{R} \) should be taken as the “driving force” for all \( H_2O \) mass transport in a frozen porous medium.

To the best of our knowledge, the above analysis leads to the first prediction of rates of pore-ice motion. O'Neill and Miller (1985, p. 286-87), who explicitly considered pore-ice motion in their numerical frost-heave simulations, avoided the need for an explicit physical model predicting the value of \( U (\mathbf{V}_i) \) in their notation) by using certain mass-balance considerations. This procedure, although not addressing the basic physics of the pore-ice regulation process, was very convenient for computational purposes. Our formulation does point out the fundamental physical controls on the regulation process but is unfortunately not convenient for computational simulations because of the idealizations involved in describing the packing of “grains”. Further work along these lines may lead to a more useful theoretical formulation. Such work would need to include a more realistic model of the pore space, which in a real porous material would contain both ice and water (e.g. O'Neill and Miller, 1985).

DISCUSSION

The foregoing analysis has two important consequences. Prediction of the transition radius \( R_t \) as a function of temperature and applied load (or drag) provides clear guidance to future investigators who may seek to extend earlier experimental work on wire regulation at sub-freezing temperatures. Such work would need to include a more realistic model of the pore space, which in a real porous material would contain both ice and water (e.g. O'Neill and Miller, 1985).

The formal analysis also points to the reasonableness of the rigid-ice approximation used by R.D. Miller and co-workers in their studies of frost-heaving. The notion that pore ice should be “rigid”, yet mobile, in a freezing soil has not gained general acceptance among workers in that field, in spite of what we view as highly persuasive arguments in its favor: arguments summarized in the recent paper by O'Neill and Miller (1985). Our analysis essentially predicts that pore-ice motion should occur in an undeformable porous medium, and specifies the functional dependence of the rate of motion on parameters such as temperature, temperature gradient, ice–pressure gradient, and grain-size. The model should be essentially valid even for a soil, which clearly is not undeformable (i.e. the soil grains might move relative to each other by processes other than ice-lens formation), as long as the rate of relative grain motion is small compared to the rate of pore-ice movement.

The analysis developed here for ice movement through a porous material should also be useful for examining theoretically the way in which the basal ice of a glacier might “invade” the pores of an underlying layer of glacial till. We defer such discussion to a separate paper, in which we examine this issue for both temperate and cold-based glaciers.

ACKNOWLEDGEMENTS

Discussions with B. Hallet on the physics of ice-lensing motivated the analyses presented above. B. Hallet, K. Hutter, and an anonymous reviewer carefully critiqued an earlier version of this paper. Correspondence with R.D. Miller helped clarify certain aspects of the pore-ice regulation problem. F. Bardsley drafted the figures and Quaternary Research Center staff helped prepare the typescript. Financial support was provided by U.S. National Science Foundation grant EAR83-19119.

REFERENCES


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**APPENDIX A**

**LIQUID-LAYER THICKNESS**

Equations (21) and (22) of the main text may be written in dimensionless form as:

\[
-(1 + h')^2 \frac{\alpha}{c} \frac{d h}{d \theta} = \left( \frac{U}{V} \right) \sin \theta - \frac{A}{R} \sin \theta + \frac{C}{G} \frac{U}{V} \sin \theta + \frac{A}{2R} \left( 1 + h' \right)^2 \sin \theta, \quad (A-1)
\]

\[
-(1 + h')^2 \frac{\alpha}{c} \frac{d h}{d \theta} = \frac{A}{R} \sin \theta (1 + h')^2 \sin \theta + \frac{C}{G} \frac{U}{V} \sin \theta + \frac{A}{2R} \left( 1 + h' \right)^2 \sin \theta, \quad (A-2)
\]

where all symbols are as defined in the main text. The very lowest-order approximation to use in integrating these equations, and one valid only for \( h' \ll 1 \), is to take \( 1 + h' \approx 1 \), which makes the right-hand sides of Equations (A-1) and (A-2) independent of \( h' \). This leads to the expressions given by Equations (30) and (31) of the main text.

The next level of approximation for \( h' \ll 1 \) would be a linearized expansion with, for example, \((1 + h')^2 \approx 1 + 3h'\). Integration of Equations (A-1) and (A-2), in this case, obviously leads to complicated exponentials (if we neglect the non-linear terms) and makes determination of the constant \( A \) a complicated exercise. The resulting expression must then be expanded in terms of trigonometric functions.
functions in order to calculate the drag (cf. Equation (33) of main text). This "refined" expression for the drag does not differ greatly from the expression found from the simpler analysis. The simplicity of Equation (35) for the drag is considered adequate reason to stay with the lowest-order analysis.

APPENDIX B

CALCULATION OF THE APPARENT ICE VISCOSITY

We have assumed ice to have a Newtonian-viscous rheology, in spite of the proliferation of experimental evidence to the contrary, in order to facilitate our analysis, much as Nye (1969, 1970) did in his glacier-sliding theory. Nonetheless, we may roughly account for the actual rheology of ice by treating the ice viscosity as stress-dependent. In particular, we will assume (cf. Shreve, 1984, p. 344, table I)

\[ \eta_i = \frac{\eta_0 e^{Q/RT_a}}{\langle \tau^2 \rangle} \]  

(B-1)

where \( \eta_0 \) = constant, \( Q \) = activation energy for creep, \( R \) = gas constant, and \( T_a \) = absolute temperature, and the meaning of \( \langle \tau^2 \rangle \) will be explained shortly.

The "effective shear stress" \( \tau \) is defined by (e.g. Paterson, 1981, p. 85)

\[ 2\tau^2 = \sum_i \tau_i^j \tau_i^j \]  

(B-2)

where the \( \tau_i^j \) are the deviatoric stresses in the ice, and the summation convention for repeated subscripts is implied. For the case of ice flow past a lubricated sphere, we found that the only non-zero stress component was \( \sigma_{rr} \), hence

\[ 2\tau^2 = \tau_{rr}^2 \tau_{rr} \]  

(B-3)

or

\[ 2\tau^2 = (\sigma_{rr}^2 - \rho)^2 \]  

(B-4)

where \( \rho \) is the mean stress, equal to \( \sigma_{rr} / 3 \) in this case.

We now assume that, in order to characterize the effective viscosity for ice deformation adjacent to the sphere, we may use \( \langle \tau^2 \rangle \), the mean-square value of \( \tau \) over the sphere’s surface:

\[ \langle \tau^2 \rangle = \frac{1}{\pi} \int_0^\pi \tau^2(\tau = R) \, d\theta \]

\[ = \frac{1}{2\pi} \int_0^\pi (\sigma_{rr} - \rho)^2 \, d\theta. \]  

(B-5)

However, it is easy to show that

\[ (\sigma_{rr} - \rho) = (1/2)\rho_d \cos \theta, \]  

(B-6)

Using Equation (B-6) in (B-5) and integrating, we find

\[ \langle \tau^2 \rangle = (1/16)\rho_d^2 \]  

(B-7)

and finally using Equation (B-7) in (B-1),

\[ \eta_i = \frac{16\eta_0 e^{Q/RT_a}}{\rho_d^2}. \]  

(B-8)

To fix the constant \( \eta_0 \), we follow Shreve and assume a viscosity of 1 bar a \((3.16 \times 10^{12} \text{ Pa s})\) at 0°C and an effective shear stress of 1 bar (0.1 MPa). Taking \( Q = 6 \times 10^4 \text{ J mol}^{-1} \) (cf. Shreve, 1984) and \( R = 8.314 \text{ J mol}^{-1} \text{ K}^{-1} \), we therefore find \( \eta_0 = 1.06 \times 10^{11} \text{ Pa s} \). Calculated values of \( \rho_d \) are then found using Equation (B-8) for ice viscosity.

MS. received 8 January 1986 and in revised form 7 April 1986