ABSTRACT. A stress–strain relation for dry snow in Greenland and Antarctica was derived. When this relation is integrated, it gives snow density as a function of time. For given surface density, temperature, and accumulation, the age of snow layers can be obtained as a function of depth in the snow-pack. Calculations compare well with observations. With some knowledge of the temperature range in the upper layer of the snow-pack, calculation for density versus depth can also be improved over the results where such temperature information was not used.

INTRODUCTION

The densification of snow has been studied by a number of investigators, including the pioneering works of Bader (1960, 1963), Benson (1962), and Anderson and Benson (1963). These authors calculated the density of dry snow for different depths. A study of the age of dry snow as a function of depth was done recently by Herron and Langway (1980), with good results, by treating the snow as two layers.

In this paper we try to answer the questions: (1) Can an age–depth relation for dry snow as one continuous layer be derived without use of an empirical formula for the density–time relation? (2) What kind of stress–strain relation is required to arrive at the steady-state density profile described by Ling (1985)?

Let us first consider snow at constant accumulation and constant temperature. Then, the process of snow densification is invariant with respect to the snow surface (Bader, 1960). If we want to know what the density of the snow-surface layer would be 10 years from now, all we have to do is to dig into the snow-pack and find the layer that is 10 years old and measure its density.

THE STRESS–STRAIN RELATION

Consider the steady-state profile equation for dry snow from Ling (1985), where \( z \) is measured positive downward from the surface,

\[
 p = p_0 + (\rho_m - p_0)(1 - e^{-z/L}).
\]  

(1)

This is based on a non-linear differential equation relation between the change of pressure \( p \) and the change of density \( \rho \):

\[
 dp^2 = c(\rho_m - \rho)dp 
\]

(1a)

where \( p \) is the density at depth \( z \), \( p_0 \) is the surface density of snow, \( \rho_m \) is the maximum attainable density, and \( L \) is a characteristic length scale, set equal to 38 m according to Ling (1985). Equation (1a) is obtained by (1) generalization of a simple physical model \( dp = c(\rho_m - \rho)dp \) to the form \( dp^2 = c(\rho_m - \rho)dp \), and (2) by setting \( n = 2 \) for mathematical convenience, as in Ling (1985). Differentiating Equation (1) with respect to \( z \) yields

\[
 \frac{dp}{dz} = \frac{1}{L}(\rho_m - p_0)e^{-z/L}.
\]  

(2)

For any snow layer of infinitesimal thickness \( dz \), the mass per unit area is

\[
 A \rho_w dt = \rho dz.
\]

(3)

where \( A \) is the accumulation rate in m/year of water equivalent, \( \rho_w \) is the density of water, and \( dt \) is the infinitesimal difference between the time a particle at the bottom of the layer was deposited and the time a particle at the top was deposited. Using Equations (1) and (3), and integrating with the assumption of constant accumulation rate \( A \), one gets

\[
 A\rho_w = \rho_m^2 + L(\rho_m - p_0)[e^{-z/L} - 1].
\]

(4)

Equation (4) gives snow age as a function of snow depth, for constant temperature. Combining Equations (1), (2), and (3) to eliminate \( z \) yields

\[
 \frac{dp}{dt} = A\rho_w/\rho L(\rho_m - \rho).
\]

(5)

Integrating Equation (5), from \( p = \rho_0 \) at time \( t = 0 \) yields

\[
 p - \rho_0 + \rho_m \ln(\rho_m - \rho)/(\rho_m - p_0) = -A\rho_w/\rho L.
\]

(6)

Now the pressure at a layer, taking \( A \) to be constant in time, is the total weight per unit area above this layer, which can be obtained by multiplying Equation (3) by \( g \) and then integrating from \( t = 0 \) to \( t = t \),

\[
 p = A\rho_w g t.
\]

(7)

where \( t = 0 \) is the time when the particle was at the surface. Substituting Equation (7) into Equation (5), to relate density to pressure, yields the stress–strain relation

\[
 (1/p)(dp/dt) = (\rho_m - \rho)p/(p^2g L).
\]

(8)

This indicates that the deformation rate of dry snow is proportional to the pressure or overload as well as to the factor \((\rho_m - \rho)\) and is inversely proportional to the time \( t \). When \( t = 0 \), the deformation rate will be infinite unless \( p = 0 \). This sudden load on a layer of snow causes infinite strain has been shown from results of work done by Colbeck and others (1978), and Costes (1963). Therefore,
such behavior in the equation is quite desirable and realistic.

Substituting Equation (6) into Equation (8) to eliminate \( t \) now gives

\[
\frac{1}{\rho} \frac{dp}{dt} = \frac{\rho_w \rho_m - \rho}{L^2 g} \rho^2 \left[ \frac{\rho_m \rho_m - \rho_0}{\rho_m - \rho} - (\rho - \rho_0) \right]
\]

(9)

This is the stress-strain relation for dry snow which we will call a "non-linear stress-strain relation" since it is derived from \( dp/\rho dt \). However, the usual way of writing the stress-strain relation for dry snow, as used by Bader (1960), Melior (1964), and Anderson (1976), among others, is of the form

\[
\frac{dp}{\rho} = \frac{\rho_m \rho_m - \rho_0}{\rho_m - \rho} - (\rho - \rho_0)
\]

(10)

Re-arranging Equation (9) to integrate from \( \rho = \rho_0 \) at time \( t = 0 \), gives

\[
F = \left[ \frac{\rho_m \rho_m - \rho_0}{\rho_m - \rho} (\rho - \rho_0) \right] dp = \frac{\rho_w \rho_m}{L^2 g} \rho^2 \int_0^\rho \left[ \frac{\rho_m \rho_m - \rho_0}{\rho_m - \rho} - (\rho - \rho_0) \right] dp.
\]

If the density ratio \( r(\rho) = \rho/\rho_m \) is introduced, from which \( dp = \rho_m dr \), the first integral of Equation (10) may be written as

\[
F = \frac{\rho_w \rho_m}{L^2 g} \rho^2 \int_{\rho_0}^{\rho} r(r) \left[ (1 - r) - \ln(1 - r) \right] dr.
\]

in which \( \rho_0 \) indicates \( r(\rho_0) \). If the change of variable \( \theta(\rho) = (1 - r) - \ln(1 - r) \) is made, from which \( dr = (1 - r) d\theta(\rho) \), it becomes

\[
F = \frac{\rho_w \rho_m}{L^2 g} \rho^2 \theta(\rho) \left| \theta(\rho) - \theta(\rho_0) \right| = \frac{\rho_w \rho_m}{L^2 g} \rho^2 \left[ (1 - r) - \ln(1 - r) \right] dr.
\]

in which \( \rho_0 \) indicates \( r(\rho_0) \). If the change of variable \( \theta(\rho) = (1 - r) - \ln(1 - r) \) is made, from which \( dr = (1 - r) d\theta(\rho) \), it becomes

\[
F = (1/2)\rho_m^2 \left[ \theta(\rho) - \theta(\rho_0) \right] d\theta.
\]

The inverse function \( r(F) \) must be obtained if \( F \) is to be determined from a known value of the second integral of Equation (10), which is equal to \( F \). Thus, given a value of \( F \), first the inverse of Equation (11) is used to get \( r \), and then \( \rho \) is recovered from \( r \) by using the relation \( r = \rho/\rho_m \). The dependence of \( r \) on \( F \) is shown in Figure 1a for selected values of the parameter \( r_0 \).

APPROXIMATING THE INVERSE OF THE DENSITY INTEGRAL

Because an inverse of Equation (11) has not been found, it is not possible to get \( r \) directly from any particular value of \( F \). Instead, a simple empirical approximation is devised to get an estimate \( \hat{r} \) of \( r \), given a value of \( F \). It is constructed simply by matching a mathematical function closely to points on the inverse function \( r(F) \).

![Graph showing the inverse of density integral](image)

*Fig. 1a. Inverse of density integral, given by Equation (10), for indicated values of the parameter \( r_0 \). The curves are tangent to the \( F = 0 \) axis at \( r = r_0 \) and asymptotically approach \( r = 1 \) as \( F \to \infty \).

b. Error curves, for indicated values of the parameter \( r_0 \), arising from use of Equation (11) to approximate the inverse of the density integral. Each curve, whose coefficients are given in Table I, minimizes the maximum error \( |\hat{r} - r| \) over the interval \( r_0 \leq r \leq 1 \) at each end of which the error is identically zero.*

While constructing the approximation, points \( r(F) \) are obtained numerically. As Equation (11) readily gives \( F \) for any value of \( r \), it is possible to search for that value of \( r \) corresponding to a particular desired \( F \). In practice, it would be tedious to undertake such an interation any time one wanted to get \( r \) from \( F \), so the approximating function is constructed as a reasonably accurate and very convenient way of estimating it.

The form chosen here for \( r(F) \) has exactly the correct behavior at the ends of the interval: \( \hat{r} = r_0 \) when \( F = 0 \), and \( \hat{r} = 1 \) when \( F \to \infty \). The normalization on \( \rho_m \) is extended here by considering the ratio \( F/\rho_m^2 \) instead of \( F \) itself. The devised function is

\[
\hat{r} = r_0 + \frac{F/\rho_m^2}{a + (F/\rho_m^2)}
\]

(12)

The empirically determined values of the two coefficients \( a \) and \( b \) are chosen to minimize the maximum value of \( |\hat{r} - r| \) occurring over the interval \( r_0 \leq r \leq 1 \). Just as \( r_0 \) is a parameter of the function \( F(r) \), it is also a parameter of its approximation. Thus, the minimizing values of \( a \) and \( b \) depend on the value of \( r_0 \) (Table I). The error \( |\hat{r} - r| \) is shown as a function of \( r \) in Figure 1b for selected values of the parameter \( r_0 \). Other functional forms could be used to approximate the inverse function more accurately, but at the cost of greater algebraic and computational complexity.

| Parameter \( r_0 \) | Coefficients | Maximum \( |\hat{r} - r| \) over \( r_0 \leq r \leq 1 \) |
|------------------|--------------|----------------------------------|
| 0.10             | 0.4382       | 0.2644                           |
| 0.15             | 0.4340       | 0.2836                           |
| 0.20             | 0.4389       | 0.3006                           |
| 0.25             | 0.4455       | 0.3162                           |
| 0.30             | 0.4620       | 0.3305                           |
| 0.35             | 0.4781       | 0.3438                           |
| 0.40             | 0.4965       | 0.3562                           |
| 0.45             | 0.5165       | 0.3679                           |

TABLE I. THE OPTIMIZING COEFFICIENTS OF THE F-INVERSE APPROXIMATION, EQUATION (11)
The accuracy of the approximation would be improved if the requirements of exact behavior at the end points were relaxed. For the case $r_0 = 0.1$, relaxing the $r^* = 1$ condition at $F = m$, by replacing the factor $1 - r_0$ by a third free coefficient, would reduce the maximum error $|r^* - r|$ by 8%; relaxing the $r^* = 1$ condition at $F = 0$ as well, by replacing the $r_0$ term by a fourth free coefficient, would reduce it by an additional 18%. Also for the $r_0 = 0.1$ case, shortening to $r_0 < r < r_{\text{max}}$ the interval over which the maximum error $|r^* - r|$ is to be minimized, by using different values for the two coefficients $a$ and $b$, would reduce the maximum error by 15% for $r_{\text{max}} = 0.7$, by 3% for 0.8, and by none for 0.85 or greater. Because the error is already of the order of density-measuring error, and because achieving those small error reductions would impair the convenience of using the approximation, the values of the coefficients given in Table I are used here with the simple form given by Equation (12); the end-point conditions are met, and the maximum error is minimized over the full interval $r_0 < r < 1$.

### THE TEMPERATURE CORRECTION

Equation (10) is for dry snow at constant temperature; but the effects of temperature variation on the densification of snow can be quite substantial, and therefore should be included. The snow temperature varies annually in the upper 10 m so that values significantly higher than the mean value, $T_0$, are encountered each year. Bader (1963) corrected for this by calculating an equivalent temperature, $T_e$, which is higher than $T_0$, because the higher temperatures in the annual variations are most effective in modifying the rates of densification in the top 10 m. We use the temperature factor $B = B_0 \exp(-E/RT)$ as in Glen (1955), Bader (1963), Anderson (1976), and Herron and Langway (1980). One may write $(1/p)(dp/dz) = B_0 \exp(-E/RT)$ but $(1/p)(dp/dt) = p$ as in Equation (9). Thus $(1/p)(dp/dt) = p \exp(E/RT)$ to include the temperature effect, we need only multiply $p$ by $B_0 \exp(-E/RT)$ on the right-hand side of Equation (10). When $T_0 = T_m$, $B_1 = 1$, so $B_1 = \exp(E/RT_m)$ and $B = \exp(E/RT_m - E/RT)$. Here $E$ is the activation energy, which is taken to be $1.33 \times 10^5$ J/mole according to Glen (1955). The gas constant $R$ is taken to be $8.3 J/Kmole$. $T$ is the temperature of the snow, and $T_m$ is the temperature at the depth of 10 m. At this depth the annual variation in temperature is less than 0.5 deg.

We assume a temperature distribution as follows:

$$T = T_m + \Delta T \exp\left[-\left[(T_m/T) - 1\right]^{1/2}\right]$$

based on the work of Woller and Schwertfeger (1968), and Benson and Bembr (1962), where $\Delta T$ is the annual temperature amplitude (that is, half the total annual range) at the snow surface of a specific station, assumed to be 15 deg in this work, and $T$ is the period, taken here to be 1 year, and $\alpha$ is the thermal diffusivity which is equal to $k/pc$ where $k$ is the thermal conductivity of snow and $c$ is the specific heat. The thermal conductivity of snow varies from 0.04 to $1.23 J/m.s^{1/2}$ for snow up to 2.5 m in depth, as shown by Lame (1985) in his work on measurements in the Antarctic. Here, we use a representative value of $k = 0.4 J/m.s^{1/2}$ and $c = 0.967 J/kg.K^{-1}$ for the upper 10 m of snow. Now, $T_0 = 0.4 Mg.m^{-3}$, and $\alpha = 1.064 J/m^2.s^{1/2}$. Thus, Equation (10), along with Equation (11), with the inclusion of the temperature factor, is

$$\rho(T) = \rho_0 \left[1 - \frac{T}{T_m}\right]^{1/2} = \frac{\rho_0 A}{L_g} \int_0^T \exp(E/RT_m) - (E/RT)dt$$

where $E$ is as given above. The $t$ coordinate is chosen so that $t = 0$ in the middle of the year when the surface layer was deposited, and this is done uniformly for any specific layer during the calculation. Depth (z) is always available as a measured parameter. Here too, for Equation (13), we need depth in terms of time. This can be approximated by using Equation (4). After expanding

$$e^{-z/(2L)}$$

into a Taylor series and keeping the first three terms:

$$A\rho_0 = \rho_0 + \left(\rho_m - \rho_0\right)z^2/(2L).$$

Therefore,

$$\rho = \rho_0 + \left[\left(\rho_m - \rho_0\right)z^2/(2L)\right]$$

Equation (12) is used to get $\rho$ from the right-hand side of Equation (13) and the corresponding depth, from Equation (3), is

$$z = \int_0^T \frac{dt}{\rho}$$

where the density distribution as a function of $t$ is obtained from Equation (13).

### RESULTS

Equations (13) and (14) have been used to calculate the age of snow layers at five stations - Crete, Site 2, and Milcent of Greenland; and Byrd Station and Little America V, Antarctica. Table II shows the input data used for the five stations. The relations between observed age and depth

<table>
<thead>
<tr>
<th>Station</th>
<th>Age (a)</th>
<th>Depth (m)</th>
<th>Temperature (K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Site 2, Greenland</td>
<td>0.4</td>
<td>0.358</td>
<td>249.7</td>
</tr>
<tr>
<td>Byrd Station, Antarctica</td>
<td>0.15</td>
<td>0.366</td>
<td>247.0</td>
</tr>
<tr>
<td>Milcent, Greenland</td>
<td>0.50</td>
<td>0.360</td>
<td>251.0</td>
</tr>
<tr>
<td>Little America V, Antarctica</td>
<td>0.221</td>
<td>0.360</td>
<td>249.0</td>
</tr>
<tr>
<td>Crete, Greenland</td>
<td>0.265</td>
<td>0.360</td>
<td>243.0</td>
</tr>
</tbody>
</table>

for the five stations are from Herron and Langway (1980), and Gow (1968). Figure 2 shows comparisons of calculated age with given depth with observed data, with and without the temperature correction. Figure 3 shows comparisons of calculated density-depth curves with observation for Byrd Station and Little America V (the two stations for which tabulated density-depth data are available to the authors), with and without the temperature correction. There does not seem to be much difference for the age versus depth calculation where we include the temperature correction; however, there is some improvement for the calculation of the density versus depth when we include the temperature correction, as shown in Figure 3, even though for Byrd Station the agreement is still not very satisfactory in the upper 20 m (Fig. 3a).

The lack of agreement in the upper 20 m results because we have ignored the discontinuity in physical properties which occurs at a porosity of about 40%, i.e. a density of 0.55 Mg m$^{-3}$ (Benson, 1962; Anderson and Benson, 1963). The significance of the change in predominant densification mechanisms at this "critical density" in Antarctic snow was emphasized by Robertson and Bentley (1975) in their study of seismic velocity gradients. Equation (12) was also used in the above calculations; the approximations are so close to the exact calculations that the differences are negligible (see Table I for the maximum error). Thus, the approximation in Equation (12) can be used with confidence in actual calculations.
CONCLUSION

We have developed a stress-strain relation for dry snow in Greenland and Antarctica. It is used to calculate the age–depth relation of dry snow, and the results are quite encouraging when given the accumulation rate, surface-snow density, and deep-core temperature. The average error with
with the observed age is less than 3%, while the maximum error is about 7.5%.

With temperature correction for the top 10 m of snow, we are able to improve on density–depth calculations.

We believe this method will be valid only in the case of dry snow in the cold regions. However, it could be extended for dry snow in other regions by changing parameters such as the characteristic length L.

When one wants to study wet snow or snow under strong thermodynamic influences such as melting and freezing, perhaps the method has to be modified or a totally different approach might be needed.

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